A rate of convergence for the homogenization limit of fully non linear equations

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Fully non linear equations arise in optimal control and game theory.

A typical problem would be:

Under Dirichlet boundary data find a function u such that

$$\sup_{\alpha} L_{\alpha} u = 0$$

with L_{α} a family of constant coefficient operators.

For instance

$$F(D^2u) = \max\left(u_{xx} + u_{yy}, \frac{1}{2}u_{xx} + 2u_{yy}\right) = 0$$

or

$$F(D^2u) = \sup_{a \in A} a_{ij} D_{ij} u = 0$$

where

$$A = \{\text{matrices with eigenvalues between 1 and } L\}$$

This is the Pucci extremal operator, and u can be described as satisfying

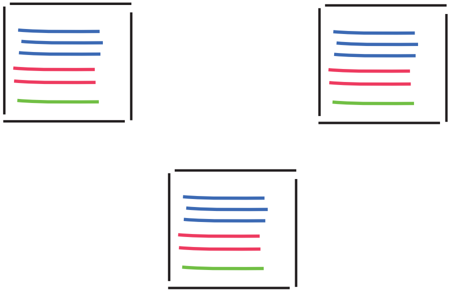
$$\sum_{\lambda_j < 0} \lambda_j + L \sum_{\lambda_j > 0} \lambda_j = 0$$

A variable Pucci operator would be, for instance

$$\sum_{\lambda_j<0} \lambda_j + L(x) \sum_{\lambda_j>0} \lambda_j = 0$$

For the homogenization setting, we will have a "family of media $L_\omega(x)$ that appear with some frequency", i.e. $\omega \in M$ a probability space (a family of equations $F(D^2u, x, \omega)$).

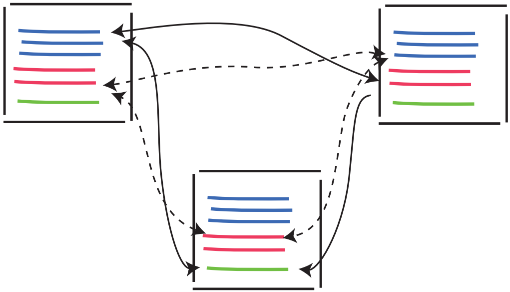
The different equations have the same frequency no matter where we stand:



(For any integer translation y, there is a measure preserving transformation τ_y such that

$$F(D^2u, x+y, \omega) = F(D^2u, x, \tau_y(\omega))$$

But they mix:



(If $\mu(A) < 1$,

$$\mu(\bigcap_{y_k} \tau_{y_k} A) \to 0$$

as the translations y_k cover the space.)

Homogenization theorem

If you look from further away, all equations become the same:

The solutions u_ε of $F(D^2 u_\varepsilon, \frac{x}{\varepsilon}, \omega)$ converge to the solution u_0 of

$$\overline{F}(D^2 u_0) = 0$$

where there is no dependence on x anymore.

Rates of convergence

The question you ask next is: Are there circumstances under which we could estimate the rate of convergence of the u_ε to u_0?

I.e., given δ, can we say that for an $\varepsilon(\delta)$ predicted, $u_\varepsilon(x, \omega)$ would be δ away from u_0, except for a set of ω's of measure δ?

This could happen only if $\mu(\bigcap_{y \in y_0} \tau_y(A))$ would go to zero at some fast rate as y_0 covers the space.

If the operators do not mix, i.e., it takes a lot of time for the blues and the reds to mix, there will be, at large scales, solutions of "only blues" and of "only reds".

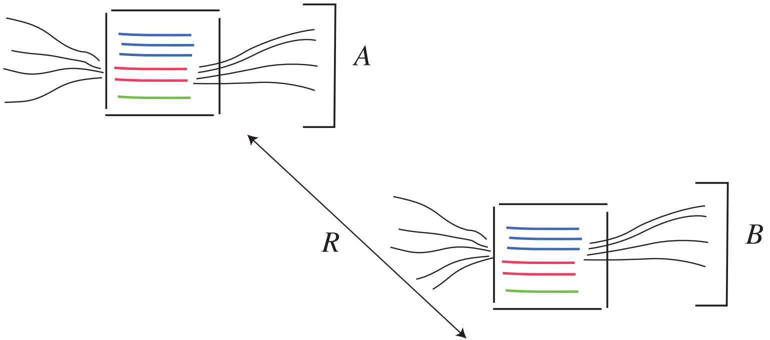
On the other hand, if blues and reds mix at a consistent rate, the picture will become "uniformly purple", i.e., we hope to be able to estimate, for a given small (epsilon) scale, how many solutions are close to the homogenization limit.

That would happen, for instance if the distributions of ω's at y_k and y_ℓ are independent: (a checkerboard)

In that case

$$\mu(\cap\, \tau_{y_j} B_j) = \prod \mu(B_j)^n$$

A more relaxed hypothesis will be "correlation decay".



$$|\mu(A \cap B) - \mu(A)\mu(B)| \leq g(R)$$

Theorem (C-Souganidis))

If the rate of decay is 3^{-k} *for* $r = 3^{k^2}$ *the rate of convergence is also* 3^{-k} *for* $\varepsilon = 3^{-k^2}$.

Note that the rate of convergence is very slow, but also the rate of decay of correlations is very slow.

This seems to happen because the diffusion process of a fully non linear equation may be much slower than a linear one (with constant coefficients).

Main facts needed for the method

1) Solutions and differences of solutions to a FNL equation satisfy an “elliptic equation’ with bounded measurable coefficients”.

$$
\begin{aligned}
0 &= F(D^2u_1, x) - F(D^2u_2, x) = \\
&= \underbrace{F_{ij}(M, x)}\, \cdot D^2(u_1, u_2)
\end{aligned}
$$

The derivative of F at an intermediate matrix

In particular, for such solutions we have

a) Harnack inequality and interior C^α (Krylov-Safanov)
b) ABP
c) Fabes-Strook

ABP If $Lu = f$, and $u \le 0$ on ∂B_1

$$\sup_{B_1} u \le C\|f\|_{L^n}$$

Fabes-Strook (A converse to ABP)

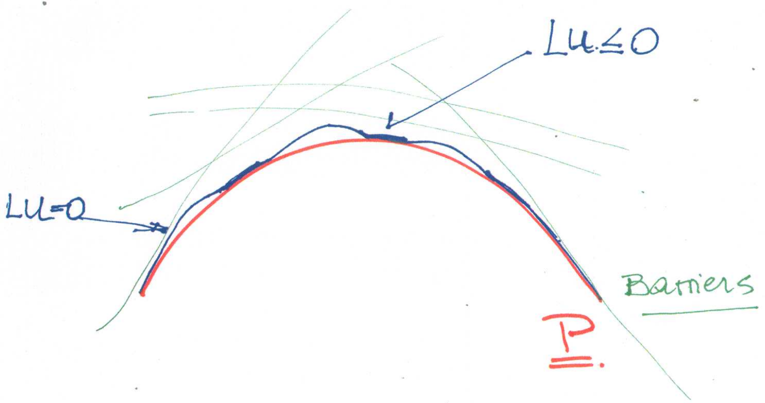
$$\text{If } \; Lu = f \le 0\,, \quad \text{and } \; u \ge 0 \; \text{ on } \; \partial B_1$$

$$u \ge \|f\|_{L^\infty}^{1-M} \|f\|_{L^n}^{M} \; \ldots \; \text{on } \; B_{1/2}$$

Remark

Technically, the slow rate of convergence we obtains seems due to the different homogeneities between ABP and Fabes-Strook above.

The obstacle problem: Given an operator L_1 (with a comparison principle) and an "obstacle" (for us a polynomial P) in a domain D (for us a cube or a ball), we will consider the function u, the smallest supersolution of $Lv \leq 0$, among those v's above P (Perron's method).


Lu≤0
Lu=0
Barriers
P.

Properties:

a) If L has the Harnack inequality and $u(x_0) = P(x_0)$

$$(u - P)(x) \leq C|x - x_0|^2$$

(Quadratic detachment)

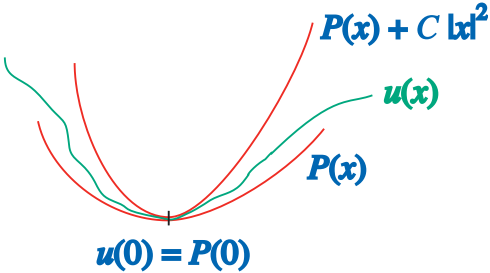
b) $Lu = L(P)\chi_{u=P}$ = bounded and negative

(no distribution across interphase)

c) If $Lv = 0$, $v = P$ on ∂D,

$$0 \leq u - v \leq C\|Lu\|_{L^n} = \|LP\chi_{u=P}\|_{L^n} \leq C|\{u = P\}|$$

a) and b)

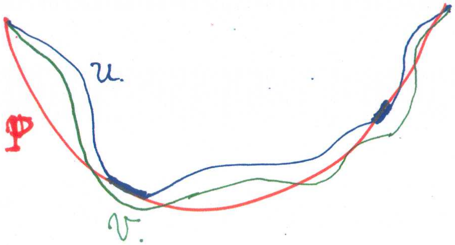


a) Quadratic separation at every seal implies

b) $F(D^2u)$ carries *no* distribution on

$$\partial\{u = P\} = \partial\Lambda \quad (\text{so } \; Lu = LP\chi_{u=P})$$

c) The mass of the contact set controls the separation between u and the "free solution" v:



$$P - v \leq u - v \leq \|LP\chi_{u=P}\|_{L^p} \leq C|\{u = P\}|$$

Proof of existence of effective equation and homogenization limit:

1) How to guess the limiting equation?
   (viscosity solution method)

A uniformly elliptic equation

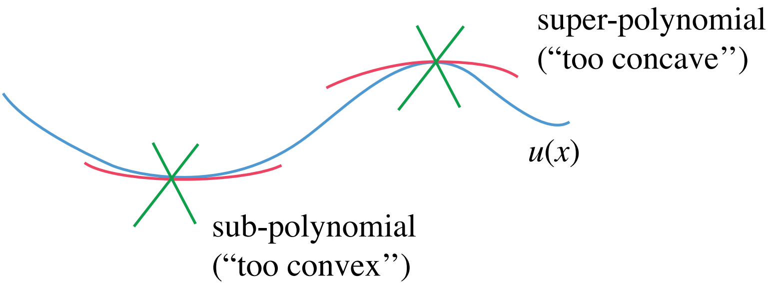
$$F(D^2u)$$

is simply a function $F(M)$ in the space of matrices, monotone in a cone of directions around the identity:
(If $N^+ \gg N^-$, $F(M+N) > F(M)$.)

In particular, $F(M) = 0$ is a Lipschitz surface $\sum$ in $\mathbb{R}^{n\times n}$ and a way to determine it would be to "list" all matrices above and below $\sum$, i.e., all quadratic polynomials that are "sub" or "super" solutions of $F(D^2P) = 0$.

(To define the Lapalacian, I would need a "long" list of all sub and super harmonic polynomials.)

Once you know this “list” of polynomials, a continuous function $u(x)$ is defined to be a “viscosity solution” of the equation $F(D^2u) = 0$ if no “sub polynomial” may touch it (locally) by below and no “super polynomial” by above.



(A continuous function would be declared “harmonic” if no subharmonic polynomial could locally touch it by below nor superharmonic by above. Note that no “touching by a polynomial” at any particular point is required.)

The remarkable fact is that such function is the unique, as regular as possible solution of $F(D^2u) = 0$.

Therefore, in order to find the effective equation and homogenization limit our main problem is to decide, give a quadratic polynomial, P, if it is going to be a sub- or super-solution of the effective equation.

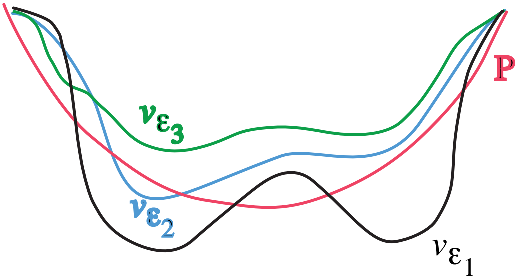
That means the following:

We fix P, and start to solve, for $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 \cdots$, v_ε solution of $F(D^2 v_\varepsilon, \chi/\varepsilon, \cdots) = 0$.

If a.s. in ω, v_ε becomes bigger than P, we declare P a *subsolution* of $\overline{F}(D^2)$.

If smaller, P should be a *supersolution*. (P can touch v_ε by above, resp. by below.)

If neither happens, *no homogenization*.



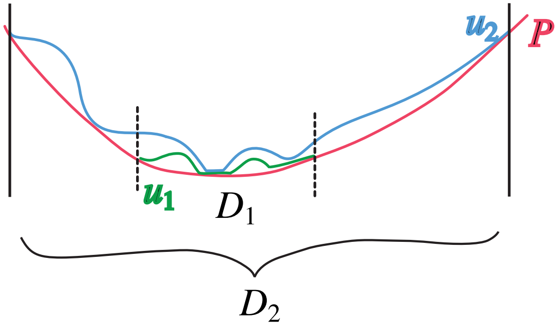
So we switch to the solution u_ε of the obstacle problem, and instead of studying the behavior of u_ε in the unit ball, we rescale by $\frac{1}{\varepsilon}$, so we work with a fixed equation $F(D^2u, x, \omega)$ in a large $(B_{1/\varepsilon})$ domain.

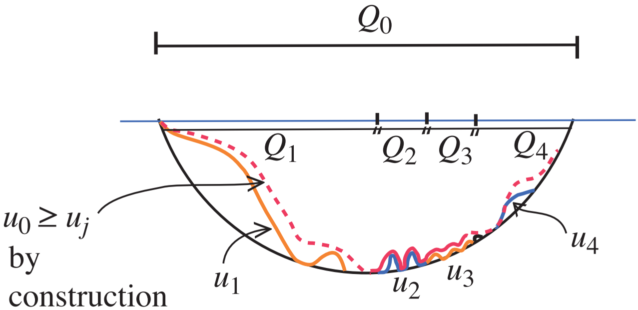
This has the advantage of

a) Compare successive solutions (in larger and larger domains)

b) The measure of the contact set or total mass of F become subadditive quantities.

u_2 is in D_1 an admissible supersolution, and bigger than u_1, the *least* supersolution.





$$|\{u_0 = P\}| \leq \Sigma\, |\{u_j = P\}|$$

$\lambda_p(Q_0,\omega) = |\{u_Q = P\}|$ is subadditive

(replaces the Birkhoff property)

$\lambda_p(Q, \omega)$ is a subadditive translation invariant quantity

$$(\lambda(Q(x + y, \omega) = \lambda(Q(x, \tau_y \omega))$$

and then,

$$\frac{\lambda(Q_R, \omega)}{|Q_R|} \xrightarrow[R \to \infty]{} \lambda_0 \text{ (a constant, a.s. in } \omega)$$

$$\text{Two cases} \quad \begin{cases} \lambda_0 = 0 \\ \lambda_0 > 0 \end{cases}$$

If $\lambda_0 = 0$, and we rescale Q_R back to Q_1 (and u to u_ε ($\varepsilon = \frac{1}{R}$))

$$\frac{\lambda(Q_R, \ )}{|Q_R|} \text{ becomes } |\{u_\varepsilon = P\}| \text{ in } Q_1$$

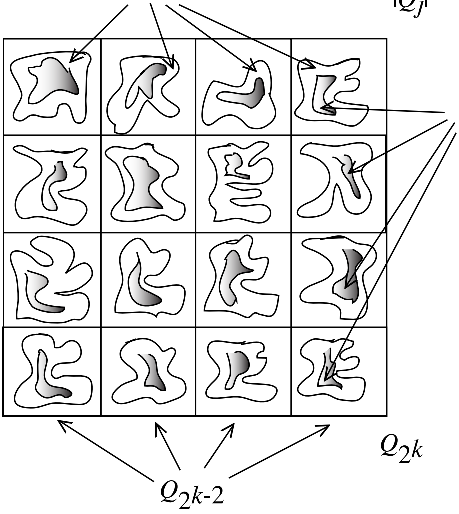
So $\lambda_0 = 0$ means that $|\{u_\varepsilon = P\}| \to 0$.

Then $|v_\varepsilon - u_\varepsilon| \to 0$ and v_ε aligns *above* P.

P should be a subsolution.

If $\lambda_0 > 0$: P must be a supersolution of $\overline{F}$.

$\{u_j = P\}$ as $k \to \infty$ $\quad \dfrac{|\{u_j = P\}|}{|Q_j|} \longrightarrow h > 0$



Portions of $u_0 = P$,
$\{u_0 = P\} \subset \cup \; \{u_j = P\}$
but, as $k \to \infty$, also

$$\frac{|\{u_0 = P\}|}{|Q_{2k}|} \longrightarrow h > 0$$

That forces $\{u_0 = P\}$ to spread all over. From the quadratic separation at every scale, $v \leq P$ at the $\varepsilon = 0$ limit.

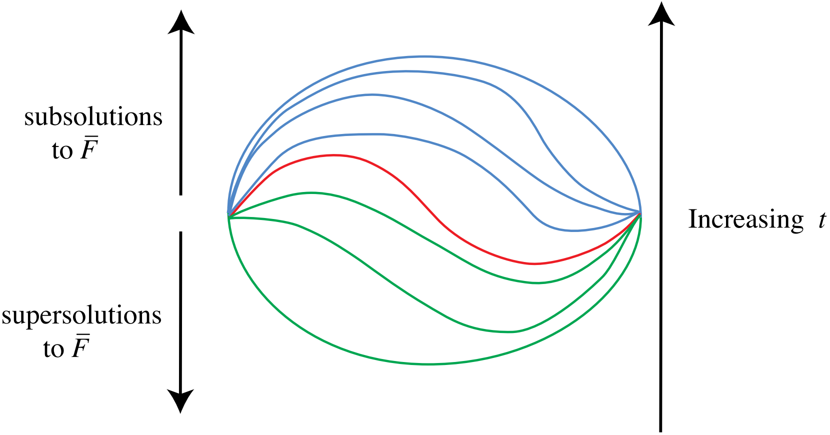
Note that implicit in the proof we construct approximate correctors to the polynomials.

At this point the existence of a homogenization limit is through viscosity solution methods:

We look at an “essential inf” (“sup”) in ω of the limiting u_ε and show that they are “super” and “sub” solutions of $\overline{F}$.
Since the super is below the sub, they must be equal.

Rate of convergence

Let us start by pointing out that we may, starting from a polynomial P_0 continuously change the polynomial to $P_t = P_0 + t(|x|^2 - 1)$ and see what happens with $\overline{F}(P_t)$ and λ_0 both for the upper obstacle (least supersolution above P_t or lower obstacle (upper subsolution below P_t).


subsolutions
to $\bar{F}$
supersolutions
to $\bar{F}$
Increasing t

Furthermore, as t separates from zero we have $(P_t - P_0)v \subset t$ and thus P_t also separates from the approximate correctors v_ε (solutions of $F(D^2 v_\varepsilon, \frac{x}{\varepsilon}))$ that are converging to P_0.

Then for u_ε^t the solution to the P_t obstacle problem (from above for t negative, from below for t positive) "$|u_\varepsilon^t - v_\varepsilon| \geq t$" a.s. as $\varepsilon \longrightarrow 0$.

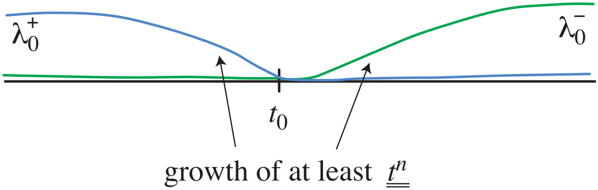
Therefore, from A-B-P theorem

$$\text{“}|\{u_\varepsilon^t = P_t\}| \geq t^n\text{”} \quad \text{a.s. as} \quad \varepsilon \to 0$$

That is:

$$\text{the ergodic limit} \quad \lambda_0^{+\text{ or }-} \geq t^n$$

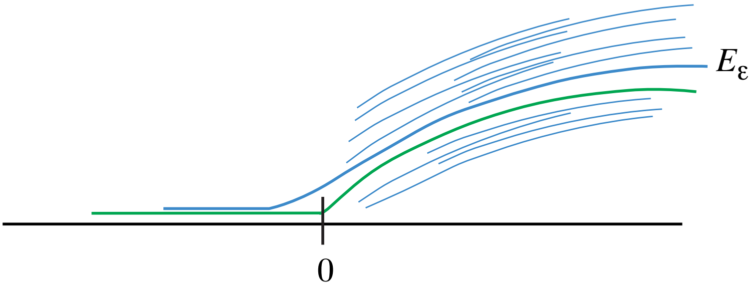
i.e., we have



In fact, since the process was subadditive for a fixed ε, the expectation E_ε of

$$|\{u_\varepsilon^t = P_t\}| \quad \left(\text{or of } \int F(D^2u, \frac{x}{\varepsilon})\right)$$

is bigger than $\lambda_0(P_t)$.



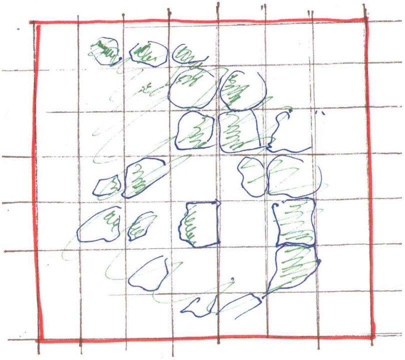
In fact, the rate of convergence of v_ε to P is clearly related to how fast the $\lambda_\varepsilon^\pm(\omega)$ converges to zero in their respective intervals since each one of them quantifies, from the A-B-P theorem how close is v_ε to P from either side.

If we go back to the picture that describes the case $\lambda_0 > 0$, and we assume, for simplicity, independence of the distribution for disjoint large squares:

Two scales: $M_0 = 2^{k_0}, M_1 = 2^{k_1}, \quad k_1 > k_0.$

- Green "blurb", contact set for M_1, contained in the union of small blue blurbs, corresponding to the 2^{k_0} cubes. The mass of the "Green blurb" for u_1^+ *or* u_1^- controls by below $(u_1^+ - u_1^-)$ (*Fabes-Strook*).

- Blue "blurbs" being independent are "well spread", and many blue blurbs for u_0^+ and u_0^- will be non empty in the same cube.



If we adjust the relation between k_0 and k_1 properly (so that 2^{k_0} is tiny with respect to 2^{k_1}) and the mass of the green blurb is not too small with respect to the large cube (of size 2^{k_1}), we have that:

Fabes-Strook versus quadratic separation implies that u_1^+, u_1^- cannot both touch the same 2^{k_0} cube.

In particular, when passing from the union of the blue blurbs to the green blurb, a fraction of the mass will be “wipped out” (those cubes with overlapping masses).

The different homogeneities of ABP and Fabes-Strook force $k_1 = Ck_0^2$ for a geometric decay on the mass and the corresponding rate of convergence.